

# Two-dimensional $\mathcal{N} = (2, 2)$ Lattice Gauge Theories with Matter in Higher Representations

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**Anosh Joseph**

*John von Neumann Institute for Computing NIC, Platanenallee 6, 15738 Zeuthen,  
GERMANY*

*Deutsches Elektronen-Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, GERMANY*

*E-mail: anosh.joseph@desy.de*

**ABSTRACT:** We construct two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric gauge theories on a Euclidean spacetime lattice with matter in the two-index symmetric and anti-symmetric representations of  $SU(N_c)$  color group. The method of twisting is used to construct such theories in the continuum and then the geometric discretization scheme is used to formulate them on the lattice. The lattice theories obtained this way are gauge-invariant, free from fermion doubling problem and exact supersymmetric at finite lattice spacing.

**KEYWORDS:** Field Theories in Lower Dimensions, Lattice Quantum Field Theory, Supersymmetric Gauge Theory, Extended Supersymmetry.

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## 1. Introduction

Supersymmetric Yang-Mills (SYM) theories are interesting classes of theories by themselves. They also serve as starting points for constructions of many phenomenologically relevant models. These theories can come with nonperturbative sectors that are less tractable analytically. Having a nonperturbative formulation of supersymmetric gauge theories would certainly advance our goal toward understanding their rich structure. Supersymmetric gauge theories constructed on a Euclidean spacetime lattice would provide a first principle approach to study the nonperturbative regimes of these theories. For certain classes of SYM theories with extended supersymmetries there are two approaches readily available to us for the constructions of their supersymmetric cousins. They are called the methods of twisting and orbifolding [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. They both give rise to identical lattices<sup>1</sup>.

Supersymmetric lattices have been constructed for several classes of SYM theories [2, 3, 4, 5, 6] including the well known  $\mathcal{N} = 4$  SYM theory<sup>2</sup>. There have been a few extensions of these formulations by incorporating matter fields in the adjoint and fundamental representations of the color group [29, 30, 31, 32, 33]. Some of them have also been extended to incorporate product gauge groups, resulting in supersymmetric quiver gauge theories on the lattice [32, 33, 34].

In this paper, we construct two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric lattice gauge theories with matter fields transforming in higher representations of  $SU(N_c)$  color group. These theories are constructed using the following procedure. We begin with a two-dimensional Euclidean SYM theory possessing eight supercharges. Such a theory can be obtained from dimensionally reducing the six-dimensional Euclidean  $\mathcal{N} = 1$  SYM. The

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<sup>1</sup>There also exist other complementary approaches to the problem of lattice supersymmetry [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

<sup>2</sup>Several aspects of this theory have been explored numerically in the recent past [23, 24, 25, 26, 27, 28].

fields and supersymmetries of this theory are then twisted to obtain a continuum theory, which is compatible with lattice discretization. Next step is to extend this theory such that it becomes a supersymmetric quiver gauge theory with two nodes and with gauge group  $SU(N_c) \times SU(N_f)$ . This can be achieved by replicating the continuum twisted theory and then changing the group representation of an appropriate subset of the field content of the theory from adjoint to the product representations  $(R, \bar{R})$  and  $(\bar{R}, R)$ , with  $R$  being the desired two-index representation and  $\bar{R}$  the corresponding complex conjugate representation. The adjoint fields of the quiver theory live on the nodes while the fields in the product representations live on the links connecting the nodes of the quiver. To construct two-dimensional  $\mathcal{N} = (2, 2)$  lattice gauge theories with two-index matter, we freeze the theory on one of the nodes of the quiver and also an appropriate set of matter fields linking the two nodes. After this restriction we have an  $SU(N_c)$  gauge theory containing matter fields in the two-index representation and with  $SU(N_f)$  flavor symmetry. Note that such a restriction of the fields is not in conflict with supersymmetry. Such constructions have been carried out by Matsuura [32] and also in Ref. [33] to formulate lattice gauge theories with matter fields in the fundamental representation. The continuum theories constructed this way, with two-index matter, can then be placed on the lattice using the method of geometric discretization. The resultant lattice theories contain adjoint fields living on the p-cells of a two-dimensional square lattice and two-index matter fields living on the sites of the same lattice. The lattice theories constructed this way enjoy gauge-invariance, freedom from fermion doublers and exact supersymmetry at finite lattice spacing.

This paper is organized as follows. In Sec. 2 we write down the twisted action and scalar supersymmetry transformations of the two-dimensional eight supercharge SYM with gauge group  $SU(N_c)$ . In Sec. 3 we construct the two-dimensional  $\mathcal{N} = (2, 2)$  gauge theory with matter fields transforming in the two-index (symmetric and anti-symmetric) representations of  $SU(N_c)$ . Our construction gives rise to matter fields with  $SU(N_f)$  flavor symmetry. We construct the lattice theory with two-index matter in Sec. 4 using the method of geometric discretization. We end with discussion and comments in Sec. 5.

## 2. $\mathcal{N} = (2, 2)$ Theories with Adjoint Matter

We begin with writing down the twisted action of the two-dimensional Euclidean SYM theory with eight supercharges. The twisted action of the theory is obtained by decomposing the fields and supercharges of the original theory under a new rotation group called the twisted rotation group [35]. In our case, the twisted rotation group is the diagonal subgroup of the product of the two-dimensional Euclidean Lorentz rotation group  $SO(2)_E$  and the  $SO(2)$  subgroup of the R-symmetry group of the original theory. The action of the two-dimensional theory is

$$S = S_{\text{SYM}}^{\mathcal{N}=(2,2)} + S_{\text{adj matter}}, \quad (2.1)$$

where the first piece is the action of the two-dimensional  $\mathcal{N} = (2, 2)$  SYM

$$S_{\text{SYM}}^{\mathcal{N}=(2,2)} = \frac{1}{g^2} \int d^2x \text{Tr} \left( -\bar{\mathcal{F}}_{mn} \mathcal{F}_{mn} + \frac{1}{2} [\bar{\mathcal{D}}_m, \mathcal{D}_m]^2 - \chi_{mn} \mathcal{D}_m \psi_n - \eta \bar{\mathcal{D}}_m \psi_m \right), \quad (2.2)$$

and the second piece contains the matter part in the adjoint representation

$$S_{\text{adj matter}} = \frac{1}{g^2} \int d^2x \text{Tr} \left( -2(\overline{\mathcal{D}}_m \bar{\phi})(\mathcal{D}_m \phi) + [\overline{\mathcal{D}}_m, \mathcal{D}_m][\bar{\phi}, \phi] + \bar{\eta} \mathcal{D}_m \bar{\psi}_m \right. \\ \left. + \bar{\chi}_{mn} \overline{\mathcal{D}}_m \bar{\psi}_n - \eta[\bar{\phi}, \bar{\eta}] - \psi_m[\phi, \bar{\psi}_m] - \frac{1}{2} \chi_{mn}[\bar{\phi}, \bar{\chi}_{mn}] + \frac{1}{2}[\bar{\phi}, \phi]^2 \right). \quad (2.3)$$

Here  $m, n = 1, 2$  and  $g$  is the coupling parameter of the theory. All fields are transforming in the adjoint representation of the  $\text{SU}(N_c)$  gauge group<sup>3</sup>.

After twisting, the fermions and supercharges transform as integer spin representations of the twisted rotation group. The fermionic degrees of freedom of the twisted theory are p-forms with  $p = 0, 1, 2$ . They are labeled as  $\{\eta, \psi_m, \chi_{mn}, \bar{\eta}, \bar{\psi}_m, \bar{\chi}_{mn}\}$ . Similarly, the twisted supercharges can also be packaged as a set of p-forms. The untwisted theory contains four scalars. After twisting, two of the scalars combine to form a two-dimensional vector  $B_m$  under the twisted rotation group. Since there are two vector fields in the twisted theory,  $A_m$  and  $B_m$ , and they both transform the same way under the twisted rotation group, it is natural to combine them to form a complexified gauge field, which we label as  $\mathcal{A}_m$ , and write down the twisted theory in a compact way. Thus the twisted theory contains two complexified gauge fields

$$\mathcal{A}_m \equiv A_m + iB_m, \quad \overline{\mathcal{A}}_m \equiv A_m - iB_m. \quad (2.4)$$

Such a construction leads to complexified covariant derivatives in the theory. They are defined by

$$\mathcal{D}_m = \partial_m + [\mathcal{A}_m, \quad] = \partial_m + [A_m + iB_m, \quad], \quad (2.5)$$

$$\overline{\mathcal{D}}_m = \partial_m + [\overline{\mathcal{A}}_m, \quad] = \partial_m + [A_m - iB_m, \quad]. \quad (2.6)$$

The complexification of gauge fields also results in complexified field strengths

$$\mathcal{F}_{mn} = [\mathcal{D}_m, \mathcal{D}_n] \quad \text{and} \quad \overline{\mathcal{F}}_{mn} = [\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n]. \quad (2.7)$$

The twisted theory also contains two scalars  $\phi$  and  $\bar{\phi}$ .

Among the twisted supercharges, the scalar (0-form) supercharge  $\mathcal{Q}$  is important for us. This supercharge is nilpotent,  $\mathcal{Q}^2 = 0$ . It does not produce any infinitesimal translations and thus we can transport this subalgebra of the twisted supersymmetry algebra to the lattice. In the twisted supersymmetry algebra one also finds that the momentum is the  $\mathcal{Q}$ -variation of something, which makes plausible the statement that one can write the energy-momentum tensor, and the entire action in a  $\mathcal{Q}$ -exact form. Thus a lattice action constructed in a  $\mathcal{Q}$ -exact form is trivially invariant under the scalar supercharge. In summary, we can use the process of twisting to construct a lattice action that respects at least one supersymmetry exact on the lattice. The lattice theories formulated using twisted

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<sup>3</sup>For the adjoint representation we use the anti-hermitian basis formed by the  $\text{SU}(N_c)$  generators in the fundamental representation, the  $N_c \times N_c$  matrices  $T^a$  with  $a = 1, 2, \dots, N_c^2 - 1$ . They have the normalization  $\text{Tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}$ . In this paper we express all group theoretical weights in terms of the defining representation.

fermions are free from the fermion doubling problem, owing to the property that the twisted fermions are geometric in nature (p-forms) and thus they can be mapped one-to-one on to the lattice from continuum [36, 37, 38, 39].

The scalar supercharge generates the following nilpotent supersymmetry transformations on the twisted fields

$$\mathcal{Q}\mathcal{A}_m = \psi_m, \quad \mathcal{Q}\bar{\mathcal{A}}_m = 0, \quad (2.8)$$

$$\mathcal{Q}\phi = \bar{\eta}, \quad \mathcal{Q}\bar{\phi} = 0, \quad (2.9)$$

$$\mathcal{Q}\eta = d, \quad \mathcal{Q}\bar{\eta} = 0, \quad (2.10)$$

$$\mathcal{Q}\psi_m = 0, \quad \mathcal{Q}\bar{\psi}_m = \bar{\mathcal{D}}_m\bar{\phi}, \quad (2.11)$$

$$\mathcal{Q}\chi_{mn} = -[\bar{\mathcal{D}}_m, \bar{\mathcal{D}}_n], \quad \mathcal{Q}\bar{\chi}_{mn} = 0, \quad (2.12)$$

$$\mathcal{Q}d = 0, \quad (2.13)$$

where  $d$  is an auxiliary field introduced for the off-shell completion of the algebra. It obeys the constraint  $d = \sum_m [\bar{\mathcal{D}}_m, \mathcal{D}_m] + [\bar{\phi}, \phi]$ . Note that while writing down the  $\mathcal{Q}$ -exact action in Eq. (2.1) we integrated out the auxiliary field.

### 3. $\mathcal{N} = (2, 2)$ Theories with Two-index Matter

We are interested in constructing two-dimensional  $\mathcal{N} = (2, 2)$  lattice gauge theories coupled to matter fields transforming in the two-index representations of  $SU(N_c)$  gauge group. To construct such theories we follow the procedure similar to the one given in Ref. [32]. We begin with the action of the theory given in Eq. (2.1). The next step is to make two copies of the theory to construct a quiver gauge theory containing two nodes and with gauge group  $SU(N_c) \times SU(N_f)$ . The nodes are labeled by  $N_c$  and  $N_f$ . Each node contains a copy of the two-dimensional  $\mathcal{N} = (2, 2)$  SYM, with gauge group  $SU(N_c)$  for the  $N_c$ -node and  $SU(N_f)$  for the  $N_f$ -node. In the quiver theory, the representation of the matter fields has now changed from adjoint to two-index and they transform as  $(R, \bar{R})$  and  $(\bar{R}, R)$  under the product gauge group. Here  $R$  is the desired two-index (symmetric or anti-symmetric) representation and  $\bar{R}$  the corresponding complex conjugate representation. The matter fields now live on the links connecting the nodes of the quiver. Next step is to freeze one of the nodes of the theory, say the  $N_f$ -node and make one set of the link fields non-dynamical by hand. The resulting theory is a two-dimensional  $\mathcal{N} = (2, 2)$  gauge theory with matter in two-index representation of  $SU(N_c)$  gauge group and with  $SU(N_f)$  flavor symmetry. The theory with two-index matter still contains the scalar supercharge and it can be discretized on a lattice. In the lattice theory, the fields of the SYM multiplet are placed on the p-cells of the lattice. The matter fields of the lattice theory occupy the sites of the lattice. Such a prescription leads to a gauge-invariant lattice theory preserving one supersymmetry exact on the lattice.

The action of the continuum theory with two-index matter has the following form

$$S = S_{\text{SYM}}^{\mathcal{N}=(2,2)} + S_{\text{2I matter}}, \quad (3.1)$$

where the first part is the same as the one given in Eq. (2.2). It contains the set of fields  $\{\mathcal{A}_m, \bar{\mathcal{A}}_m, \eta, \psi_m, \chi_{mn}\}$  transforming in the adjoint representation of  $\text{SU}(N_c)$ . The matter part of the action contains fields transforming in the adjoint, two-index and two-index complex conjugate representations of  $\text{SU}(N_c)$  and is given by

$$\begin{aligned} S_{2\text{I matter}} = & \frac{1}{g^2} \int d^2x \text{Tr} \left( 2\bar{\phi}^\alpha (\bar{\mathcal{D}}_m \mathcal{D}_m) \phi^\alpha - [\bar{\mathcal{D}}_m, \mathcal{D}_m] (\phi^\alpha \bar{\phi}^\alpha) + \bar{\psi}_m^\alpha \mathcal{D}_m \bar{\eta}^\alpha \right. \\ & + \bar{\chi}_{mn}^\alpha \bar{\mathcal{D}}_m \bar{\psi}_n^\alpha + \eta \bar{\eta}^\alpha \bar{\phi}^\alpha - \psi_m \phi^\alpha \bar{\psi}_m^\alpha + \frac{1}{2} \chi_{mn} \bar{\chi}_{mn}^\alpha \bar{\phi}^\alpha \\ & \left. + \frac{1}{2} (\phi^\alpha \bar{\phi}^\alpha)^2 + \frac{1}{2} (\bar{\phi}^\alpha \phi^\alpha)^2 \right), \end{aligned} \quad (3.2)$$

with  $\alpha$  an index labeling the  $\text{SU}(N_f)$  flavor symmetry.

The set of fields  $\{\phi^\alpha, \bar{\eta}^\alpha, \bar{\chi}_{12}^\alpha\}$  transform in the two-index representation  $\text{R}$  and the set of fields  $\{\bar{\phi}^\alpha, \bar{\psi}_1^\alpha, \bar{\psi}_2^\alpha\}$  transform in the complex conjugate representation  $\bar{\text{R}}$ .

For a generic field  $\Psi$  in the two-index representation the action of the covariant derivative is given by

$$\mathcal{D}_m \Psi^{\text{R}} = \partial_m \Psi^{\text{R}} + \mathcal{A}_m^{\text{R}} \Psi^{\text{R}}, \quad (3.3)$$

while for a field  $\bar{\Psi}$  in the complex conjugate representation

$$\mathcal{D}_m \bar{\Psi}^{\bar{\text{R}}} = \partial_m \bar{\Psi}^{\bar{\text{R}}} + \bar{\Psi}^{\bar{\text{R}}} \mathcal{A}_m^{\bar{\text{R}}}, \quad (3.4)$$

with  $\mathcal{A}_m^{\text{R}} = \mathcal{A}_m^a (t^a)^{\text{R}}$ ,  $\mathcal{A}_m^{\bar{\text{R}}} = \mathcal{A}_m^a (t^a)^{\bar{\text{R}}}$  and  $a = 1, 2, \dots, N_c^2 - 1$ .

The generators of the two-index representation  $(t^a)^{\text{R}}$  and  $(t^a)^{\bar{\text{R}}}$  can be constructed from the generators of the defining representation of  $\text{SU}(N_c)$  using appropriate projection operators [40]. Denoting  $(t^a)^{2\text{A}}$  and  $(t^a)^{2\text{S}}$  as the generators for the two-index anti-symmetric (2A) and symmetric (2S) representations, respectively, below we write down their expressions in terms of the generators of the fundamental representation of  $\text{SU}(N_c)$ .

Using the projection operators  $P^A = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})$ ,  $i < j$ ,  $k < l$ , and  $P^S = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$ ,  $i \leq j$ ,  $k \leq l$ , where  $i, j, k, l = 1, 2, \dots, N_c$ , we pick out the two-index anti-symmetric and symmetric parts of the  $\text{SU}(N_c)$  representation. The generator  $(t^a)^{2\text{A}}$  is defined as

$$(t^a)^{2\text{A}} = (t^a)_{ij,kl}^{2\text{A}} = \frac{1}{2}[(T^a)_{ik}\delta_{jl} - (T^a)_{jk}\delta_{il} + \delta_{ik}(T^a)_{jl} - \delta_{jk}(T^a)_{il}], \quad (3.5)$$

and the generator  $(t^a)^{2\text{S}}$

$$(t^a)^{2\text{S}} = (t^a)_{ij,kl}^{2\text{S}} = \frac{1}{2}[(T^a)_{ik}\delta_{jl} + (T^a)_{jk}\delta_{il} + \delta_{ik}(T^a)_{jl} + \delta_{jk}(T^a)_{il}]. \quad (3.6)$$

Indeed they have the correct dimensions and indices of the respective representations:  $d_{2\text{A}/2\text{S}} = \frac{1}{2}N_c(N_c \mp 1)$  and  $T^{2\text{A}/2\text{S}} = \frac{1}{2}(N_c \mp 2)$ . The generators of the complex conjugate representations are obtained from  $(t^a)^{\bar{\text{R}}} = ((t^a)^{\text{R}})^*$ .

## 4. Lattice Theories

The supersymmetric gauge theories constructed above can be discretized on a Euclidean spacetime lattice using the geometric discretization scheme formulated in Refs. [8, 41, 42]. The complexified gauge fields of the continuum theory,  $\mathcal{A}_m(x)$  are mapped to appropriate complexified Wilson links  $\mathcal{U}_m(\mathbf{n})$  defined at a location on the two-dimensional square lattice denoted by the integer vector  $\mathbf{n}$ . These link fields are associated with unit length vectors in the coordinate directions  $\hat{\nu}_m$  from the site  $\mathbf{n}$ . The components of the fermion field  $\psi_m(\mathbf{n})$  live on the same oriented links as that of their bosonic superpartners  $\mathcal{U}_m(\mathbf{n})$ . The field  $\eta(\mathbf{n})$  is placed on the site  $\mathbf{n}$ . The components of the field  $\chi_{mn}(\mathbf{n})$  are placed on a set of diagonal face links with orientation  $\mathbf{n} + \hat{\nu}_m + \hat{\nu}_n \rightarrow \mathbf{n}$ .

We can write down the gauge transformation rules for the lattice fields in the adjoint representation respecting the p-cell and orientation assignments on the lattice. For  $G(\mathbf{n}) \in \text{SU}(N_c)$ , we have the following gauge transformation prescriptions [39, 42]

$$\mathcal{U}_m(\mathbf{n}) \rightarrow G(\mathbf{n})\mathcal{U}_m(\mathbf{n})G^\dagger(\mathbf{n} + \hat{\nu}_m), \quad (4.1)$$

$$\bar{\mathcal{U}}_m(\mathbf{n}) \rightarrow G(\mathbf{n} + \hat{\nu}_m)\bar{\mathcal{U}}_m(\mathbf{n})G^\dagger(\mathbf{n}), \quad (4.2)$$

$$\eta(\mathbf{n}) \rightarrow G(\mathbf{n})\eta(\mathbf{n})G^\dagger(\mathbf{n}), \quad (4.3)$$

$$\psi_m(\mathbf{n}) \rightarrow G(\mathbf{n})\psi_m(\mathbf{n})G^\dagger(\mathbf{n} + \hat{\nu}_m), \quad (4.4)$$

$$\chi_{mn}(\mathbf{n}) \rightarrow G(\mathbf{n} + \hat{\nu}_m + \hat{\nu}_n)\chi_{mn}(\mathbf{n})G^\dagger(\mathbf{n}). \quad (4.5)$$

The covariant derivatives in the continuum are mapped to covariant difference operators on the lattice. The covariant derivatives  $\mathcal{D}_m$  ( $\bar{\mathcal{D}}_m$ ) of the continuum theory become forward and backward covariant difference operators  $\mathcal{D}_m^{(+)}$  ( $\bar{\mathcal{D}}_m^{(+)}$ ) and  $\mathcal{D}_m^{(-)}$  ( $\bar{\mathcal{D}}_m^{(-)}$ ), respectively of the lattice theory. The forward and backward covariant difference operators act on the adjoint lattice fields in the following way

$$\mathcal{D}_m^{(-)}\psi_m(\mathbf{n}) = \mathcal{U}_m(\mathbf{n})\psi_m(\mathbf{n}) - \psi_m(\mathbf{n} - \hat{\nu}_m)\mathcal{U}_m(\mathbf{n} - \hat{\nu}_m), \quad (4.6)$$

$$\mathcal{D}_m^{(+)}\psi_n(\mathbf{n}) = \mathcal{U}_m(\mathbf{n})\psi_n(\mathbf{n} + \hat{\nu}_m) - \psi_n(\mathbf{n})\mathcal{U}_m(\mathbf{n} + \hat{\nu}_m), \quad (4.7)$$

$$\bar{\mathcal{D}}_m^{(-)}\psi_m(\mathbf{n}) = \psi_m(\mathbf{n})\bar{\mathcal{U}}_m(\mathbf{n}) - \bar{\mathcal{U}}_m(\mathbf{n} - \hat{\nu}_m)\psi_m(\mathbf{n} - \hat{\nu}_m), \quad (4.8)$$

$$\bar{\mathcal{D}}_m^{(+)}\chi_{nq}(\mathbf{n}) = \chi_{nq}(\mathbf{n} + \hat{\nu}_m)\bar{\mathcal{U}}_m(\mathbf{n}) - \bar{\mathcal{U}}_m(\mathbf{n} + \hat{\nu}_m)\chi_{nq}(\mathbf{n}). \quad (4.9)$$

These expressions reduce to the corresponding continuum results for the adjoint covariant derivative in the naive continuum limit. They also transform under gauge transformations like the corresponding lattice link field carrying the same indices. Such a discretization prescription leads to a lattice action with terms corresponding to gauge-invariant closed loops.

The field strength on the lattice is given by the expression  $\mathcal{F}_{mn}(\mathbf{n}) = \mathcal{D}_m^{(+)}\mathcal{U}_n(\mathbf{n})$ . It is automatically antisymmetric in its indices and also it transforms like a 2-form on the lattice.

We also need to define the action of the covariant difference operators on the lattice fields transforming in the two-index representations. The two-index matter fields of the lattice theory live on the sites of the lattice. They transform in the two-index representations

of  $SU(N_c)$ . We have the following set of rules for the action of the covariant derivatives on two-index fields.

The covariant difference operator acts on the lattice variables in the two-index representation the following way

$$\mathcal{D}_m^{(+)}\Phi^R(\mathbf{n}) \equiv \mathcal{U}_m(\mathbf{n})\Phi^R(\mathbf{n} + \hat{\nu}_m) - \Phi^R(\mathbf{n}), \quad (4.10)$$

$$\overline{\mathcal{D}}_m^{(+)}\Phi^R(\mathbf{n}) \equiv \Phi^R(\mathbf{n} + \hat{\nu}_m) - \overline{\mathcal{U}}_m(\mathbf{n})\Phi^R(\mathbf{n}), \quad (4.11)$$

$$\mathcal{D}_m^{(-)}\Phi^R(\mathbf{n}) \equiv \mathcal{D}_m^{(+)}\Phi^R(\mathbf{n} - \hat{\nu}_m), \quad \overline{\mathcal{D}}_m^{(-)}\Phi^R(\mathbf{n}) \equiv \overline{\mathcal{D}}_m^{(+)}\Phi^R(\mathbf{n} - \hat{\nu}_m). \quad (4.12)$$

For lattice variables in the complex conjugate representation we have the following set of rules for the action of the covariant difference operator

$$\mathcal{D}_m^{(+)}\Phi^{\bar{R}}(\mathbf{n}) \equiv \Phi^{\bar{R}}(\mathbf{n} + \hat{\nu}_m) - \Phi^{\bar{R}}(\mathbf{n})\mathcal{U}_m(\mathbf{n}), \quad (4.13)$$

$$\overline{\mathcal{D}}_m^{(+)}\Phi^{\bar{R}}(\mathbf{n}) \equiv \Phi^{\bar{R}}(\mathbf{n} + \hat{\nu}_m)\overline{\mathcal{U}}_m(\mathbf{n}) - \Phi^{\bar{R}}(\mathbf{n}), \quad (4.14)$$

$$\mathcal{D}_m^{(-)}\Phi^{\bar{R}}(\mathbf{n}) \equiv \mathcal{D}_m^{(+)}\Phi^{\bar{R}}(\mathbf{n} - \hat{\nu}_m), \quad \overline{\mathcal{D}}_m^{(-)}\Phi^{\bar{R}}(\mathbf{n}) \equiv \overline{\mathcal{D}}_m^{(+)}\Phi^{\bar{R}}(\mathbf{n} - \hat{\nu}_m). \quad (4.15)$$

We see that the above expressions reduce to the corresponding continuum covariant derivatives in the naive continuum limit by letting  $\mathcal{U}_m(\mathbf{n}) = \mathbf{I} + \mathcal{A}_m(\mathbf{n}) + \dots$  and  $\overline{\mathcal{U}}_m(\mathbf{n}) = \mathbf{I} - \overline{\mathcal{A}}_m(\mathbf{n}) + \dots$ .

The fields in the two-index and its complex conjugate representations are mapped on to lattice sites, with the gauge transformations

$$\Phi^R(\mathbf{n}) \rightarrow G(\mathbf{n})\Phi^R(\mathbf{n}), \quad (4.16)$$

$$\Phi^{\bar{R}}(\mathbf{n}) \rightarrow \Phi^{\bar{R}}(\mathbf{n})G^\dagger(\mathbf{n}). \quad (4.17)$$

We also note that the method of geometric discretization maps the continuum fields on to the lattice one-to-one and thus the lattice theories constructed this way are free from fermion doubling problem [36, 37, 38, 39]. The placement and orientations of the twisted fields on the lattice respect the scalar supersymmetry and gauge symmetry of the lattice theory. The unit cell of the two-dimensional lattice theory is given in Fig. 1.

The scalar supersymmetry acts on the lattice variables the following way

$$\mathcal{Q}\mathcal{U}_m(\mathbf{n}) = \psi_m(\mathbf{n}), \quad \mathcal{Q}\overline{\mathcal{U}}_m(\mathbf{n}) = 0, \quad (4.18)$$

$$\mathcal{Q}\phi^\alpha(\mathbf{n}) = \bar{\eta}^\alpha(\mathbf{n}), \quad \mathcal{Q}\bar{\phi}^\alpha(\mathbf{n}) = 0, \quad (4.19)$$

$$\mathcal{Q}\eta(\mathbf{n}) = d(\mathbf{n}), \quad \mathcal{Q}\bar{\eta}^\alpha(\mathbf{n}) = 0, \quad (4.20)$$

$$\mathcal{Q}\psi_m(\mathbf{n}) = 0, \quad \mathcal{Q}\bar{\psi}_m^\alpha(\mathbf{n}) = \left(\overline{\mathcal{D}}_m^{(+)}\bar{\phi}^\alpha\right)(\mathbf{n}), \quad (4.21)$$

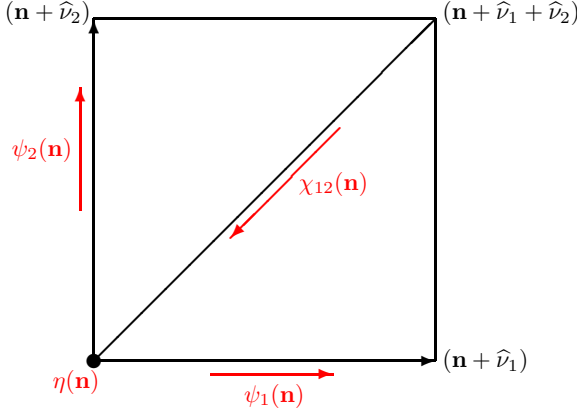
$$\mathcal{Q}\chi_{mn} = -\left(\overline{\mathcal{D}}_m^{(+)}\mathcal{U}_n\right)(\mathbf{n}), \quad \mathcal{Q}\bar{\chi}_{mn}^\alpha(\mathbf{n}) = 0, \quad (4.22)$$

$$\mathcal{Q}d(\mathbf{n}) = 0. \quad (4.23)$$

The action of the two-dimensional  $\mathcal{N} = (2, 2)$  lattice gauge theory with two-index matter has the following form

$$S = S_{\text{SYM}}^{\mathcal{N}=(2,2)} + S_{\text{2I matter}}, \quad (4.24)$$





**Figure 1:** The unit cell of the two-dimensional  $\mathcal{N} = (2, 2)$  lattice SYM with orientation assignments for the twisted fermions. The complexified bosons  $\mathcal{U}_m(\mathbf{n})$  follow the same orientations and link assignments as that of their superpartners  $\psi_m(\mathbf{n})$ .

where

$$\begin{aligned}
S_{\text{SYM}}^{\mathcal{N}=(2,2)} = \sum_{\mathbf{n}} \text{Tr} \Big\{ & \left( \bar{\mathcal{U}}_n(\mathbf{n} + \hat{\nu}_m) \bar{\mathcal{U}}_m(\mathbf{n}) - \bar{\mathcal{U}}_m(\mathbf{n} + \hat{\nu}_n) \bar{\mathcal{U}}_n(\mathbf{n}) \right) \\
& \times \left( \mathcal{U}_m(\mathbf{n}) \mathcal{U}_n(\mathbf{n} + \hat{\nu}_m) - \mathcal{U}_n(\mathbf{n}) \mathcal{U}_m(\mathbf{n} + \hat{\nu}_n) \right) \\
& + \frac{1}{2} \left( \mathcal{U}_m(\mathbf{n}) \bar{\mathcal{U}}_m(\mathbf{n}) - \bar{\mathcal{U}}_m(\mathbf{n} - \hat{\nu}_m) \mathcal{U}_m(\mathbf{n} - \hat{\nu}_m) \right)^2 \\
& + \frac{1}{2} (\delta_{mq} \delta_{nr} - \delta_{mr} \delta_{nq}) \chi_{mn}(\mathbf{n}) \left( \mathcal{U}_q(\mathbf{n}) \psi_r(\mathbf{n} + \hat{\nu}_q) - \psi_r(\mathbf{n}) \mathcal{U}_q(\mathbf{n} + \hat{\nu}_r) \right) \\
& + \eta(\mathbf{n}) \left( \psi_m(\mathbf{n}) \bar{\mathcal{U}}_m(\mathbf{n}) - \bar{\mathcal{U}}_m(\mathbf{n} - \hat{\nu}_m) \psi_m(\mathbf{n} - \hat{\nu}_m) \right) \Big\}, \tag{4.25}
\end{aligned}$$

and

$$\begin{aligned}
S_{2\text{I matter}} = \sum_{\mathbf{n}} \text{Tr} \Big( & 2 \bar{\phi}^\alpha(\mathbf{n}) \bar{\mathcal{D}}_m^{(-)} \mathcal{D}_m^{(+)} \phi^\alpha(\mathbf{n}) - \left( \bar{\mathcal{D}}_m^{(-)} \mathcal{U}_m(\mathbf{n}) \right) \left( \phi^\alpha(\mathbf{n}) \bar{\phi}^\alpha(\mathbf{n}) \right) \\
& + \bar{\psi}_m^\alpha(\mathbf{n}) \mathcal{D}_m^{(+)} \bar{\eta}^\alpha(\mathbf{n}) + \bar{\chi}_{mn}^\alpha(\mathbf{n}) \bar{\mathcal{D}}_m^{(+)} \bar{\psi}_n^\alpha(\mathbf{n}) + \eta(\mathbf{n}) \bar{\eta}^\alpha(\mathbf{n}) \bar{\phi}^\alpha(\mathbf{n}) \\
& - \psi_m(\mathbf{n}) \phi^\alpha(\mathbf{n} + \hat{\nu}_m) \bar{\psi}_m^\alpha(\mathbf{n}) + \frac{1}{2} \chi_{mn}(\mathbf{n}) \bar{\chi}_{mn}^\alpha(\mathbf{n}) \bar{\phi}^\alpha(\mathbf{n} + \hat{\nu}_m + \hat{\nu}_n) \\
& + \frac{1}{2} (\phi^\alpha(\mathbf{n}) \bar{\phi}^\alpha(\mathbf{n}))^2 + \frac{1}{2} (\bar{\phi}^\alpha(\mathbf{n}) \phi^\alpha(\mathbf{n}))^2 \Big). \tag{4.26}
\end{aligned}$$

## 5. Discussion and Comments

In this paper, we have detailed the constructions of two-dimensional  $\mathcal{N} = (2, 2)$  supersymmetric lattice gauge theories coupled with matter fields in higher representations of gauge group  $\text{SU}(N_c)$ . The process of twisting allows us to write down the theories in the continuum in a way compatible with lattice discretization. We used the method of geometric discretization to formulate these theories on a two-dimensional Euclidean spacetime lattice. The lattice theories constructed this way are gauge-invariant, doubler free and retain

one supercharge exact on the lattice. The matter fields of these theories are in the two-index symmetric and antisymmetric representations of  $SU(N_c)$  gauge group. The process of un-gauging one of the nodes of the quiver, while constructing the theory, also results in matter fields to have  $SU(N_f)$  flavor symmetry. We note that for gauge group  $SU(3)$ , 2A representation is the same as the fundamental representation. So, for  $N_c = 3$ , this is also a construction for the  $SU(3)$  theory with fundamental fermions. We could also use different projectors while constructing the lattice theory such that we have SO and Sp theories with matter in the two-index representations. We also note that one could construct  $\mathcal{N} = (2, 2)$  supersymmetric quiver lattice gauge theories with matter fields transforming in the product representations  $(R, \bar{R})$  and  $(\bar{R}, R)$  with  $R$  the two-index representation (symmetric or anti-symmetric) and  $\bar{R}$  the corresponding complex conjugate representation. One could also construct lattice theories with eight supercharges coupled to matter fields in two-index representations using the approach detailed in this paper. For such constructions the starting point would be the sixteen supercharge Yang-Mills theory in four-dimensions.

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